Sample Final Answers

1. (a) As x gets close to a, $f(x)$ gets close to L.

   (b) $f(x)$ is cts at a if
   $$\lim_{x \to a} f(x) = f(a).$$

   (c) The derivative of $f(x)$ at a is
   $$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   (d) The integral of $f(x)$ from a to b is
   $$\int_a^b f(x)\ dx = \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x f(x_i)$$
   where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

2. (a) The Fundamental Theorem of Calculus (Part I) states that if f is a continuous function, and
   $$g(x) = \int_a^x f(t)\ dt$$
   then $g'(x)$ exists and equals $f(x)$.

   (b) The Fundamental Theorem of Calculus (Part II) states that if f is a continuous function, and $g(x)$ is an anti-derivative of $f(x)$, then
   $$\int_a^b f(x)\ dx = g(b) - g(a)$$

3. (a) Dividing by the highest of power of x (namely, x^3), we get
   $$\lim_{x \to \infty} \frac{2 - x^{-1} + 3x^{-2} - 2x^{-3}}{x^{-3} + x^{-2} + 6} = \frac{2}{6} = \frac{1}{3}$$

   (b) If we substitute $x = 1$, we get $\frac{0}{0}$. So we use L'Hopital's rule to get
   $$\lim_{x \to 1} \frac{x^{10} - 1}{x^4 - 1} = \lim_{x \to 1} \frac{10x^9}{4x^3} = \frac{10}{4} = \frac{5}{2}$$

(c) If we take x to ∞, we get $\frac{\infty}{\infty}$. So we use L'Hopital's rule to get

$$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

So the limit does not exist.

(d) If we take $x = 0$, we get $\frac{0}{0}$. So we use L'Hopital's rule to get

$$\lim_{x\to 0} \frac{1-\cos x}{\sin x} = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

4. (a) We use product rule to get

$$f'(x) = (e^x)(\sin x + 2) + (e^x + 1)(\cos x)$$

(b) Using quotient rule, we get

$$f'(x) = \frac{(x^2+1)(2e^{2x}) - (2x)(e^{2x})}{(x^2+1)^2} = \frac{2e^{2x}(x^2+1-x)}{(x^2+1)^2}$$

(c) By the Fundamental theorem of calculus (Part I),

$$f'(x) = x^3 - x^2$$

5. Using chain rule, we get

$$\frac{dy}{dx} = e^{x^2+3}(2x)$$

So the slope of the tangent line at $x = 0$ is 0. Thus the slope of the normal line is $\frac{-1}{0}$, which does not exist. Therefore the normal line does not exist.

6. We differentiate both sides to get

$$\begin{aligned} 3y^2\frac{dy}{dx} - \cos y\frac{dy}{dx} &= \frac{(x^2+1)(1) - (2x)(x)}{(x^2+1)^2} \\ (3y^2 - \cos y)\frac{dy}{dx} &= \frac{-x^2+1}{(x^2+1)^2} \\ \frac{dy}{dx} &= \frac{-x^2+1}{(x^2+1)^2(3y^2 - \cos y)} \end{aligned}$$

7. Let V be the volume of the cube, A the surface area of the cube, and e the length of an edge. We know $\frac{dV}{dt}$, and we want to find $\frac{dA}{dt}$. So we need to relate V and A.

   The area of one side of the cube is e^2, so since there are six sides to a cube, the total surface area is $A = 6e^2$. Also, the volume of the cube is $V = e^3$. Thus $e = V^{1/3}$. Thus, substituting this into the formula for A gives $A = 6V^{2/3}$.

   Differentiating both sides with respect to time t, we get

   $$\frac{dA}{dt} = 6\frac{2}{3}V^{-1/3}\frac{dV}{dt} = \frac{4}{V^{1/3}}\frac{dV}{dt}$$

   We know that $e = 30$, so $V^{1/3} = e = 30$, and $\frac{dV}{dt} = 10$. Thus we have

   $$\frac{dA}{dt} = \frac{4}{(30)}(10) = \frac{40}{30} = \frac{4}{3}$$

   Thus the surface area is increasing at a rate of $\frac{4}{3}cm^2$/min.

8. Let x be the distance from the waterskier to the base of the ramp, and let h be the height of the waterskier from the water. We know $\frac{dx}{dt}$ and we want to find $\frac{dh}{dt}$.

   First, we can find the length of the ramp using Pythagoras' Theorem: it is $\sqrt{15^2 + 4^2} = \sqrt{241}$. Then by similar triangles, we know that

   $$\frac{h}{x} = \frac{4}{\sqrt{241}}$$

   Thus

   $$h = \frac{4x}{\sqrt{241}}$$

   Taking the derivative with respect to time t, we get

   $$\frac{dh}{dt} = \frac{4}{\sqrt{241}}\frac{dx}{dt} = \frac{4}{\sqrt{241}}(30) = \frac{120}{\sqrt{241}}$$

   Thus the waterskier's height is increasing at a rate of $\dfrac{120}{\sqrt{241}}$ ft/sec.

9. (a) First, we find the derivative:

$$f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$$

Since the are no points where the derivative does not exist, the only critical points occur when $f'(x) = 0$. So the critical points are $x = 0, -1, 1$.

We then test each of these critical point values, as well as the endpoints (-1 and 4). We get $f(-1) = 2, f(0) = 3, f(1) = 2, f(4) = 227$. Thus the minimum is 2, and the maximum 227.

(b) First, we find the derivative:

$$f'(x) = \frac{(x^2+4)(2x) - (2x)(x^2-4)}{(x^2+4)^2} = \frac{2x^3 + 8x - 2x^3 + 8x}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}$$

The only points where the derivative might not exist are where $x^2 + 4 = 0$. Since this never occurs, there are no points where the derivative does not exist. Thus the only critical points are when $f'(x) = 0$; this happens when $x = 0$.

We then test the critical point value, as well as the endpoints (-4 and 4). We get $f(-4) = \frac{3}{5}, f(0) = -1, f(4) = \frac{3}{5}$. Thus the minimum is -1, and the maximum $\frac{3}{5}$.

(c) First, we find the derivative:

$$f'(x) = (e^{-x}) + x(-e^{-x}) = e^{-x}(1 - x)$$

Since the are no points where the derivative does not exist, the only critical points occur when $f'(x) = 0$. So the critical points are $1 - x = 0$, so $x = 1$.

We then test this critical point, as well as the endpoints (0 and 2). We get $f(0) = 0, f(1) = e^{-1}, f(2) = 2e^{-2}$. Plugging the last two values into a calculator, one can find that $e^{-1} > 2e^{-2}$. Thus 0 is the minimum, and e^{-1} the maximum.

(d) First, we find the derivative:

$$f'(x) = \cos x - \sin x$$

Since the are no points where the derivative does not exist, the only critical points occur when $f'(x) = 0$, so when $\cos x = \sin x$ in the interval $[0, \frac{\pi}{3}]$. This only happens when $x = \frac{\pi}{4}$.

We then test this critical point, as well as the endpoints (0 and $\frac{\pi}{3}$). We get $f(0) = 1, f\left(\frac{\pi}{4}\right) = \sqrt{2}, f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}+1}{2}$. Plugging the last two values into a calculator, one can find that $\sqrt{2} > \frac{\sqrt{3}+1}{2} > 1$. Thus 1 is the minimum, and $\sqrt{2}$ the maximum.

10. (a) We begin by finding when the derivative equals 0. $y' = -2 - 3x^2$, so we want to find when $-3x^2 = 2$. Since x^2 is always positive, this never occurs. So the whole space is the only interval: $(-\infty, \infty)$. We take a test point in that interval ($x = 0$) and since $f'(0) - 2 < 0$, $f(x)$ is decreasing on the interval $(-\infty, \infty)$ (that is, everywhere). Since the function is always decreasing, there are no local maxima or minima.

    We next find where $y'' = 0$. Since $y'' = -6x$, this happens when $x = 0$. So our intervals of concavity are $(-\infty, 0), (0, \infty)$. Taking test points -1 and 1, we get $y''(-1) = 6 > 0$ and $y''(1) = -6 < 0$. So the curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$, and 0 is a point of inflection.

    (b) We begin by finding where the derivative equals 0. Using quotient rule,

$$y' = \frac{(x+8)(2x) - (x^2)(1)}{(x+8)^2} = \frac{x^2 + 16x}{(x+8)^2}$$

    So the derivative will be 0 at $x = 0$ and $x = -16$. Our intervals will be $(-\infty, -16), (-16, 0), (0, \infty)$. Taking test points $1, -1, -20$, we find $y'(1) > 0, y'(-1) < 0, y'(-20) > 0$. Thus on $(-\infty, -16)$ and $(0, \infty)$, $f(x)$ is increasing, while on $(-16, 0)$, $f(x)$ is decreasing. So $x = -16$ is a local max, and $x = 0$ is a local min.

    We now find the 2nd derivative:

$$y'' = \frac{(x+8)^2(2x+16) - (2)(x+8)(x^2+16x)}{(x+8)^4}$$

Simplifying, this becomes

$$y'' = \frac{(2x^3 + 32x^2 + 128x + 16x^2 + 256x + 1024) - (2x^3 + 16x^2 + 32x^2 + 256x)}{(x+8)^4}$$

So we have

$$y'' = \frac{128x + 1024}{(x+8)^4} = \frac{128(x+8)}{(x+8)^4}$$

So $y'' = 0$ when $x = -8$. Taking test points -9 and 0, we find that on $(-\infty, -8)$, $f(x)$ is concave down, while on $(-8, \infty)$, $f(x)$ is concave up. However, $x = -8$ is not an inflection point since the function is not defined at $x = -8$.

(c) We begin by finding where the derivative equals 0.

$$y' = e^{2x-x^2}(2 - 2x)$$

So the derivative will be 0 when $2 = 2x$, so $x = 1$. Taking test points 0 and 2, we find that $y'(0) > 0$ while $y'(2) < 0$. So on $(-\infty, 1)$, $f(x)$ increasing, while on $(1, \infty)$, $f(x)$ is decreasing. Thus $x = 1$ is a local maximum.

Taking the second derivative, we get

$$y'' = (2 - 2x)e^{2x-x^2}(2 - 2x) + (-2)(e^{2x-x^2})$$

Factoring e^{2x-x^2} gives

$$y'' = e^{2x-x^2}(4 - 8x + 4x^2 - 2) = e^{2x-x^2}(2x^2 - 4x + 1)$$

So the second derivative will be 0 when $2x^2 - 4x + 1 = 0$. We can use quadratic formula to find the solutions to this equation: $x = 1 \pm \frac{1}{\sqrt{2}}$. Taking test points $0, 1, 2$, we get $y''(0) > 0, y''(1) < 0, y''(2) > 0$. So the function is concave up on $(-\infty, 1 - \frac{1}{\sqrt{2}})$ and $(1 + \frac{1}{\sqrt{2}}, \infty)$, while it is concave down on $(1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}})$. Thus $x = 1 \pm \frac{1}{\sqrt{2}}$ are inflection points.

(d) We first find the derivative:

$$y' = 2\cos 2x$$

Thus the derivative is 0 when $\cos 2x = 0$. In the interval $[0, \pi]$, this occurs when $x = \frac{\pi}{4}$, $x = \frac{3\pi}{4}$. Taking test points $0, \frac{\pi}{2}, \pi$, we get $y'(0) > 0, y'(\frac{\pi}{2}) < 0, y'(\pi) > 0$. So the function is increasing on $[0, \frac{\pi}{4})$ and $(\frac{3\pi}{4}, \pi]$, while decreasing on $(\frac{\pi}{4}, \frac{3\pi}{4})$. Thus $x = \frac{\pi}{4}$ is a local maximum, and $x = \frac{3\pi}{4}$ is a local minimum.

The second derivative is

$$y'' = -4 \sin 2x$$

Thus the derivative is 0 when $\sin 2x = 0$. In the interval $[0, \pi]$, this occurs when $x = \frac{\pi}{2}$. Taking test points $0, \pi$, we find $y''(0) < 0$ while $y''(\pi) > 0$. Thus the function is concave down on $[0, \frac{\pi}{2})$, and concave up on $(\frac{\pi}{2}, \pi]$. Thus $x = \frac{\pi}{2}$ is an inflection point.

11. To find the horizontal asymptotes, we find the limit as x goes to ∞. So we calculate:

$$\lim_{x \to \infty} \frac{x^2 - 1}{3x^2 + 6x - 24} = \frac{1}{3}$$

So the function has a horizontal asymptote to the line $y = \frac{1}{3}$.

To find the vertical asymptotes, we find where the function goes off to ∞; namely where we divide by 0. For this function, this occurs when $3x^2 + 6x - 24 = 0$, so when $x^2 + 2x - 8 = 0$, or $(x + 4)(x - 2) = 0$. So there will be vertical asymptotes at $x = 2$ and $x = -4$.

Finally, we want to find what direction the function goes as it approaches these asymptotes (∞ or $-\infty$). As x goes to -4 from below, y is positive, so it approaches ∞. As x approaches -4 from above, y is negative, so it approaches $-\infty$. As x approaches 2 from below, y is negative, so it approaches $-\infty$. As x approaches 2 from above, y is positive, so it approaches ∞.

12. For this function, $\Delta x = \dfrac{b - a}{n} = \dfrac{5 - 1}{4} = 1$ and $x_i = a + i\Delta x = 1 + i$. So $x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 5$. So we have the Riemann sum is

$$= \sum_{i=1}^{4} \Delta x f(x_i)$$

$$
\begin{aligned}
&= (1)f(2) + (1)f(3) + (1)f(4) + (1)f(5) \\
&= (2^3 - 2) + (3^3 - 2) + (4^3 - 2) + (5^3 - 2) \\
&= 6 + 25 + 62 + 123 \\
&= 216
\end{aligned}
$$

13. (a) The general antiderivative is $e^x - 6x + C$.

    (b) The function is $f(x) = -2x^{-1/2}$, so the general antiderivative is

    $$\frac{-2x^{1/2}}{1/2} + C = -4\sqrt{x} + C$$

    (c) The function is $f(x) = x^{-1} + x^{-2}$, so the general antiderivative is

    $$\ln x + \frac{x^{-1}}{-1} + C = \ln x - \frac{1}{x} + C$$

    (d) The general antiderivative is $-3\cos x - 2\sin x + C$.

14. (a) An antiderivative of $4x + 3$ is $2x^2 + 3x$, so

    $$\int_2^8 4x + 3 \; dx = [2(8)^2 + 3(8)] - [2(2)^2 + 3(2)] = 152 - 14 = 148$$

    (b) An antiderivative of $5x^{-3}$ is $\frac{-5}{2x^2}$, so

    $$\int_{-5}^{5} \frac{5}{x^3} \; dx = \left[\frac{-5}{2(5)^2}\right] - \left[\frac{-5}{2(-5)^2}\right] = 0$$

    (c) The general antiderivative of $(4 - x)^9$ is

    $$\frac{-(4-x)^{10}}{10} + C$$

    (d) If we substitute $u = x^2 + 1$, then $du = 2x \; dx$, so $\frac{1}{2}du = x \; dx$. Thus

    $$\int \frac{x}{(x^2+1)^2} \; dx = \int \frac{1}{2u^2} \; du$$

    Integrating gives

    $$\frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C$$

(e) If we substitute $x = \sin\theta$, then $dx = \cos\theta\ d\theta$, so

$$\int \cos\theta \sin^6\theta\ d\theta = \int u^6\ du$$

Integrating gives

$$\frac{u^7}{7} + C = \frac{(\sin\theta)^7}{7} + C$$

(f) If we substitute $u = e^x + 1$, then $du = e^x\ dx$; when $x = 0$, $u = e^0 + 1 = 2$, when $x = 1$, $u = e + 1$. Thus

$$\int_0^1 \frac{e^x}{e^x + 1}\ dx = \int_2^{e+1} \frac{1}{u}\ du = \ln(e+1) - \ln(2)$$